Week 4 — Complex Numbers

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Abstract

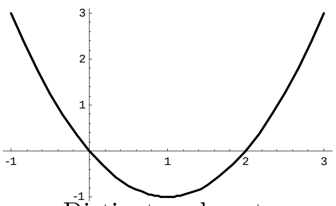
Cartesian and polar form of a complex number. The Argand diagram. Roots of unity. The relationship between exponential and trigonometric functions. The geometry of the Argand diagram.

1 The Need For Complex Numbers

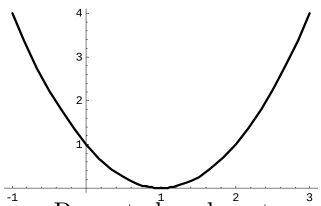
All of you will know that the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

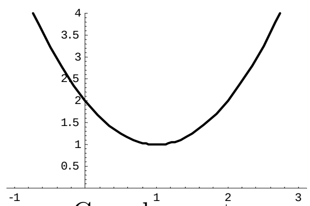
and solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2 - 4ac > 0$ then these two roots are real and distinct; graphically they are where the curve $y = ax^2 + bx + c$ cuts the x-axis. When $b^2 - 4ac = 0$ then we have one real root and the curve just touches the x-axis here. But what happens when $b^2 - 4ac < 0$? Then there are no real solutions to the equation as no real squares to give the negative $b^2 - 4ac$. From the graphical point of view the curve $y = ax^2 + bx + c$ lies entirely above or below the x-axis.



Distinct real roots



Repeated real root



Complex roots

It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2 - 4ac < 0$. During the Renaissance the quadratic would have been considered unsolvable or its roots would have been called *imaginary.* (The term 'imaginary' was first used by the French Mathematician René Descartes (1596-1650). Whilst he is known more as a philosopher, Descartes made many important contributions to mathematics and helped found co-ordinate geometry – hence the naming of Cartesian co-ordinates.) If we imagine $\sqrt{-1}$ to exist, and that it behaves (adds and multiplies) much the same as other numbers then the two roots of the quadratic can be written in the form

$$x = A \pm B\sqrt{-1} \tag{2}$$

where

$$A = -\frac{b}{2a} \quad \text{and} \quad B = \frac{\sqrt{4ac - b^2}}{2a} \quad \text{are real numbers.}$$

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But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these 'imaginary' numbers started proving so useful (especially in the work of Cauchy and Gauss) that essentially the philosophical concerns just got forgotten about.

Notation 1 *We shall from now on write i for $\sqrt{-1}$. This notation was first introduced by the Swiss mathematician Leonhard Euler (1707-1783). Much of our modern notation is due to him including e and π. Euler was a giant in 18th century mathematics and the most prolific mathematician ever. His most important contributions were in analysis (eg. on infinite series, calculus of variations). The study of topology arguably dates back to his solution of the Königsberg Bridge Problem. (Many books, particularly those written for engineers and physicists use j instead.)*

Definition 2 *A complex number is a number of the form $a + bi$ where a and b are real numbers. If $z = a + bi$ then a is known as the real part of z and b as the imaginary part. We write $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. Note that real numbers are complex – a real number is simply a complex number with no imaginary part. The term 'complex number' is due to the German mathematician Carl Gauss (1777-1855). Gauss is considered by many the greatest mathematician ever. He made major contributions to almost every area of mathematics from number theory, to non-Euclidean geometry, to astronomy and magnetism. His name precedes a wealth of theorems and definitions throughout mathematics.*

Notation 3 *We write $\mathbb{C}$ for the set of all complex numbers.*

One of the first major results concerning complex numbers and which conclusively demonstrated their usefulness was proved by Gauss in 1799. From the quadratic formula (1) we know that all quadratic equations can be solved using complex numbers – what Gauss was the first to prove was the much more general result:

Theorem 4 *(FUNDAMENTAL THEOREM OF ALGEBRA). The roots of any polynomial equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ with real (or complex) coefficients a_i are complex. That is there are n (not necessarily distinct) complex numbers $\gamma_1, \ldots, \gamma_n$ such that*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_n (x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n).$$

In particular the theorem shows that an n degree polynomial has, counting multiplicities, n roots in $\mathbb{C}$.

The proof of this theorem is far beyond the scope of this article. Note that the theorem only guarantees the *existence* of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula which plainly gives us the roots. The theorem gives no hints as to where in $\mathbb{C}$ these roots are to be found.

2 Basic Operations

We add, subtract, multiply and divide complex numbers much as we would expect. We add and subtract complex numbers by adding their real and imaginary parts:-

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)\,i, \\ (a + bi) - (c + di) &= (a - c) + (b - d)\,i.\end{aligned}$$

We can multiply complex numbers by expanding the brackets in the usual fashion and using $i^2 = -1$,

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)\,i,$$

and to divide complex numbers we note firstly that $(c + di)(c - di) = c^2 + d^2$ is real. So

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i.$$

The number $c - di$ which we just used, as relating to $c + di$, has a special name and some useful properties – see Proposition 11.

Definition 5 *Let $z = a + bi$. The conjugate of z is the number $a - bi$ and this is denoted as $\overline{z}$ (or in some books as z^*).*

- Note from equation (2) that when the *real* quadratic equation $ax^2 + bx + c = 0$ has complex roots then these roots are conjugates of each other. Generally if z_0 is a root of the polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$ where the a_i are real then so is its conjugate $\overline{z_0}$.

Problem 6 *Calculate, in the form $a + bi$, the following complex numbers:*

$$(1+3i)+(2-6i)\,, \quad (1+3i)-(2-6i)\,, \quad (1+3i)\,(2-6i)\,, \quad \frac{1+3i}{2-6i}.$$

The addition and subtraction are simple calculations, adding (and substracting) real parts, then imaginary parts:

$$\begin{aligned}(1+3i)+(2-6i) &= (1+2)+(3+(-6))\,i = 3-3i;\\ (1+3i)-(2-6i) &= (1-2)+(3-(-6))\,i = -1+9i.\end{aligned}$$

And multiplying is just a case of expanding brackets and remembering $i^2 = -1$.

$$(1+3i)\,(2-6i) = 2+6i-6i-18i^2 = 2+18 = 20.$$

Division takes a little more care, and we need to remember to multiply through by the conjugate of the denominator:

$$\frac{1+3i}{2-6i} = \frac{(1+3i)\,(2+6i)}{(2-6i)\,(2+6i)} = \frac{2+6i+6i+18i^2}{2^2+6^2} = \frac{-16+12i}{40} = \frac{-2}{5} + \frac{3}{10}i.$$

We present the following problem because it is a common early misconception involving complex numbers — if we need a new number i as the square root of -1 then shouldn't we need another one for the square root of i? But $z^2 = i$ is just another polynomial equation, with complex coefficients, and two (perhaps repeated) roots are guaranteed by the Fundamental Theorem of Algebra. They are also quite easy to calculate:–

Problem 7 *Find all those z that satisfy $z^2 = i$.*

Suppose that $z^2 = i$ and $z = a + bi$, where a and b are real. Then

$$i = (a+bi)^2 = \left(a^2 - b^2\right) + 2abi.$$

Comparing the real and imaginary parts we see that

$$a^2 - b^2 = 0 \quad \text{and} \quad 2ab = 1.$$

So $b = \pm a$ from the first equation. Substituting $b = a$ into the second equation gives $a = b = 1/\sqrt{2}$ or $a = b = -1/\sqrt{2}$. Substituting $b = -a$ into the second equation of gives $-2a^2 = 1$ which has no real solution in a.

So the two z which satisfy $z^2 = i$, i.e. the two square roots of i, are

$$\frac{1+i}{\sqrt{2}} \quad \text{and} \quad \frac{-1-i}{\sqrt{2}}.$$

Notice, as with square roots of real numbers, that the two square are negative one another.

Problem 8 *Use the quadratic formula to find the two solutions of*

$$z^2 - (3+i)\,z + (2+i) = 0.$$

We see that $a = 1$, $b = -3 - i$, and $c = 2 + i$. So

$$b^2 - 4ac = (-3-i)^2 - 4 \times 1 \times (2+i) = 9 - 1 + 6i - 8 - 4i = 2i.$$

Knowing

$$\sqrt{i} = \pm\frac{1+i}{\sqrt{2}},$$

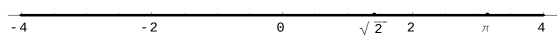
from the previous problem, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{(3+i) \pm \sqrt{2i}}{2} = \frac{(3+i) \pm \sqrt{2}\sqrt{i}}{2} \\ &= \frac{(3+i) \pm (1+i)}{2} = \frac{4+2i}{2} \quad \text{or} \quad \frac{2}{2} = 2+i \quad \text{or} \quad 1. \end{aligned}$$

Note that the two roots are not conjugates of one another – this need not be the case here as the coefficients a, b, c are not all real.

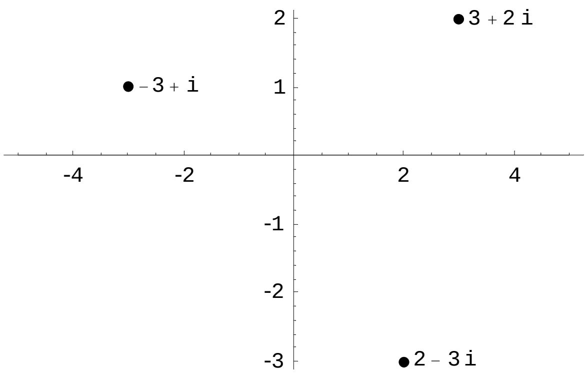
3 The Argand Diagram

The real numbers are often represented on the *real line* which increase as we move from left to right



The real number line

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed $\mathbb{C}$ is sometimes referred to as the *complex plane,* but more commonly when we represent $\mathbb{C}$ in this manner we call it an *Argand diagram.* (After the Swiss mathematician Jean-Robert Argand (1768-1822)). The point (a, b) represents the complex number $a + bi$ so that the x-axis contains all the real numbers, and so is termed the *real axis,* and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part) and so is referred to as the *imaginary axis.*



An Argand diagram

We can think of $z_0 = a + bi$ as a point in an Argand diagram but it can often be useful to think of it as a vector as well. Adding z_0 to another complex number translates that number by the vector $\binom{a}{b}$. That is the map $z \mapsto z + z_0$ represents a translation a units to the right and b units up in the complex plane.

Note that the conjugate $\overline{z}$ of a point z is its mirror image in the real axis. So, $z \mapsto \overline{z}$ represents reflection in the real axis. We shall discuss in more detail the geometry of the Argand diagram in Sections 9 to 11.

A complex number z in the complex plane can be represented by Cartesian co-ordinates, its real and imaginary parts, but equally useful is the representation of z by polar co-ordinates. If we let r be the distance of z from the origin and, if $z \neq 0$, we let θ be the angle that the line connecting z to the origin makes with the positive real axis then we can write

$$z = x + iy = r\cos\theta + ir\sin\theta. \tag{3}$$

The relations between z's Cartesian and polar co-ordinates are simple — we see that

$$\begin{aligned} x &= r\cos\theta \text{ and } y = r\sin\theta, \\ r &= \sqrt{x^2+y^2} \text{ and } \tan\theta = \frac{y}{x}. \end{aligned}$$

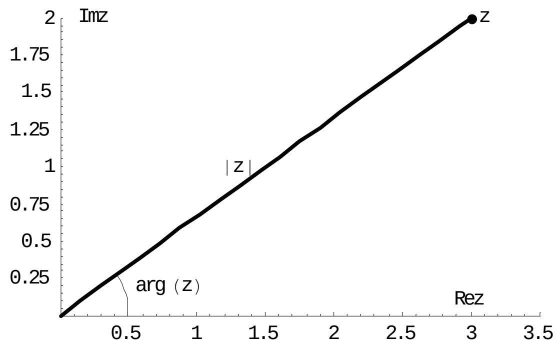
Definition 9 *The number r is called the modulus of z and is written $|z|$. If $z = x + iy$ then*

$$|z| = \sqrt{x^2+y^2}.$$

Definition 10 *The number θ is called the argument of z and is written $\arg z$. If $z = x + iy$ then*

$$\sin\arg z = \frac{y}{\sqrt{x^2+y^2}} \quad \textit{and} \quad \cos\arg z = \frac{x}{\sqrt{x^2+y^2}}.$$

Note that the argument of 0 is undefined. Note also that $\arg z$ is defined only upto multiples of 2π. For example the argument of $1+i$ could be $\pi/4$ or $9\pi/4$ or $-7\pi/4$ etc. For simplicity in this course we shall give all arguments in the range $0 \leq \theta < 2\pi$ so that $\pi/4$ would be the preferred choice here.



A Complex Number's Cartesian and Polar Co-ordinates

We now prove some important formulae about properties of the modulus, argument and conjugation:–

Proposition 11 *The modulus, argument and conjugate functions satisfy the following properties. Let $z, w \in \mathbb{C}$. Then*

$$\begin{aligned}
|zw| &= |z|\,|w|, & \left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \quad \textit{if } w \neq 0, \\
\overline{z \pm w} &= \overline{z} \pm \overline{w}, & \overline{zw} &= \overline{z}\,\overline{w}, \\
\arg(zw) &= \arg z + \arg w \ \textit{ if } z, w \neq 0, & z\overline{z} &= |z|^2, \\
\arg\left(\frac{z}{w}\right) &= \arg z - \arg w \quad \textit{if} \ z, w \neq 0, & \overline{\left(\frac{z}{w}\right)} &= \frac{\overline{z}}{\overline{w}} \quad \textit{if } w \neq 0, \\
|\overline{z}| &= |z|, & \arg\overline{z} &= -\arg z, \\
|z+w| &\leq |z| + |w|, & ||z| - |w|| &\leq |z-w|.
\end{aligned}$$

A selection of the above statements is proved here; the remaining ones are left as exercises.
Proof. $|zw| = |z|\,|w|$.
Let $z = a + bi$ and $w = c + di$. Then $zw = (ac - bd) + (bc + ad)\,i$ so that

$$\begin{aligned} |zw| &= \sqrt{(ac-bd)^2 + (bc+ad)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} = |z|\,|w|. \end{aligned}$$

■
Proof. $\arg(zw) = \arg z + \arg w$.
Let $z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\Theta + i\sin\Theta)$. Then

$$\begin{aligned} zw &= rR(\cos\theta + i\sin\theta)(\cos\Theta + i\sin\Theta) \\ &= rR((\cos\theta\cos\Theta - \sin\theta\sin\Theta) + i(\sin\theta\cos\Theta + \cos\theta\sin\Theta)) \\ &= rR(\cos(\theta+\Theta) + i\sin(\theta+\Theta)). \end{aligned}$$

We can read off that $|zw| = rR = |z|\,|w|$, which is a second proof of the previous part, and also that

$$\arg(zw) = \theta + \Theta = \arg z + \arg w, \quad \text{up to multiples of } 2\pi.$$

■
Proof. $\overline{zw} = \overline{z}\,\overline{w}$.
Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned} \overline{zw} &= \overline{(ac-bd) + (bc+ad)\,i} \\ &= (ac-bd) - (bc+ad)\,i \\ &= (a-bi)(c-di) = \overline{z}\,\overline{w}. \end{aligned}$$

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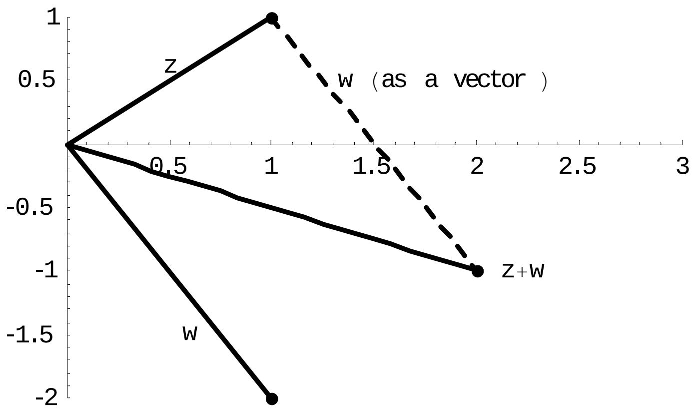
Proof. (Triangle Inequality) $|z + w| \leq |z| + |w|$ – a diagrammatic proof of this is simple and explains the inequality's name:–



A Diagrammatic Proof Of The Triangle Inequality

Note that the shortest distance between 0 and $z + w$ is the modulus of $z + w$. This is shorter in length than the path which goes from 0 to z to $z + w$. The total length of this second path is $|z| + |w|$.

For an algebraic proof, note that for any complex number

$$z+\bar{z}=2\operatorname{Re}z \quad \text{and} \quad \operatorname{Re}z\leq|z|\,.$$

So for $z,w\in\mathbb{C}$,

$$\frac{z\bar{w}+\bar{z}w}{2}=\operatorname{Re}(z\bar{w})\leq|z\bar{w}|=|z|\,|\bar{w}|=|z|\,|w|\,.$$

Then

$$\begin{aligned}
|z+w|^2 &= (z+w)\,\overline{(z+w)} \\
&= (z+w)\,(\bar{z}+\bar{w}) \\
&= z\bar{z}+z\bar{w}+\bar{z}w+w\bar{w} \\
&\leq |z|^2+2\,|z|\,|w|+|w|^2=(|z|+|w|)^2\,,
\end{aligned}$$

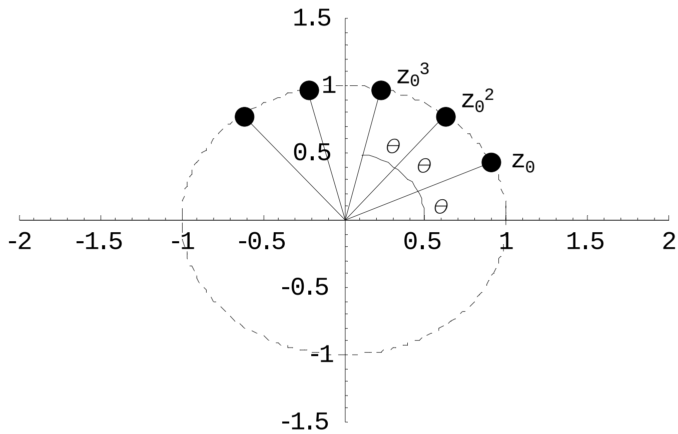
to give the required result. ∎

4 Roots Of Unity.

Consider the complex number

$$z_0=\cos\theta+i\sin\theta$$

where θ is some real number. The modulus of z_0 is 1 and the argument of z_0 is θ.



Powers of z_0

In Proposition 11 we proved for $z,w\neq 0$ that

$$|zw|=|z|\,|w| \quad \text{and} \quad \arg(zw)=\arg z+\arg w,$$

and so for any integer n, and any $z\neq 0$, we have that

$$|z^n|=|z|^n \quad \text{and} \quad \arg(z^n)=n\arg z.$$

So the modulus of $(z_0)^n$ is 1 and the argument of $(z_0)^n$ is $n\theta$ or putting this another way we have the famous theorem due to De Moivre:

Theorem 12 *(DE MOIVRE'S THEOREM) For a real number θ and integer n we have that*

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n\,.$$

(De Moivre(1667-1754), a French protestant who moved to England, is best remembered for this formula but his major contributions were in probability and appeared in his *The Doctrine Of Chances* (1718)).

We apply these ideas now to the following:

Problem 13 *Let n be a natural number. Find all those complex z such that $z^n = 1$.*

We know from the Fundamental Theorem of Algebra that there are (counting multiplicities) n solutions – these are known as *the nth roots of unity.*

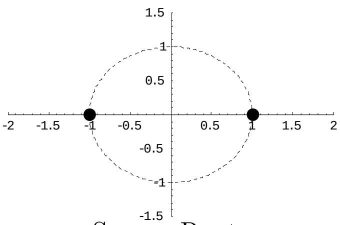
Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

- When $n = 2$ we have
$$0 = z^2 - 1 = (z - 1)(z + 1)$$
and so the square roots of 1 are ± 1.

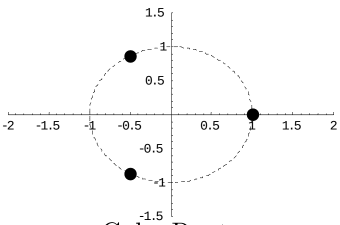
- When $n = 3$ we can factorise as follows
$$0 = z^3 - 1 = (z - 1)\left(z^2 + z + 1\right).$$
So 1 is a root and completing the square we see
$$0 = z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$
which has roots $-1/2 \pm \sqrt{3}i/2$. So the cube roots of 1 are
$$1 \text{ and } \frac{-1}{2} + \frac{\sqrt{3}}{2}i \text{ and } \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$$

- When $n = 4$ we can factorise as follows
$$0 = z^4 - 1 = \left(z^2 - 1\right)\left(z^2 + 1\right) = (z - 1)(z + 1)(z - i)(z + i),$$
so that the fourth roots of 1 are $1, -1, i$ and $-i$.

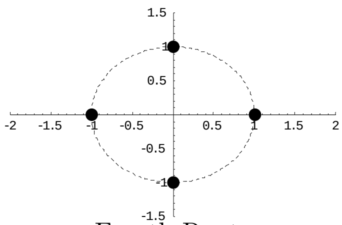
Plotting these roots on Argand diagrams we can see a pattern developing



Square Roots



Cube Roots



Fourth Roots

Returning to the general case suppose that
$$z = r(\cos\theta + i\sin\theta) \quad \text{and satisfies} \quad z^n = 1.$$
Then by the observations preceding De Moivre's Theorem z^n has modulus r^n and has argument $n\theta$ whilst 1 has modulus 1 and argument 0. Then comparing their moduli
$$r^n = 1 \implies r = 1.$$
Comparing arguments we see $n\theta = 0$ up to multiples of 2π. That is $n\theta = 2k\pi$ for some integer k giving $\theta = 2k\pi/n$. So we see that if $z^n = 1$ then z has the form
$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{where } k \text{ is an integer.}$$
At first glance there seem to be an infinite number of roots but, as cos and sin have period 2π, then these z repeat with period n.

Hence we have shown

Proposition 14 *The nth roots of unity, that is the solutions of the equation $z^n = 1$, are*

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \textit{where} \quad k = 0, 1, 2, \ldots, n-1.$$

Plotted on an Argand diagram these nth roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1.

Problem 15 *Find all the solutions of the cubic $z^3 = -2 + 2i$.*

If we write $-2 + 2i$ in its polar form we have

$$-2 + 2i = \sqrt{8}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

So if $z^3 = -2 + 2i$ and z has modulus r and argument θ then

$$r^3 = \sqrt{8} \text{ and } 3\theta = \frac{3\pi}{4} \text{ up to multiples of } 2\pi,$$

which gives

$$r = \sqrt{2} \text{ and } \theta = \frac{\pi}{4} + \frac{2k\pi}{3} \text{ for some integer } k.$$

As before we need only consider $k = 0, 1, 2$ (as other k lead to repeats) and so the three roots are

$$\begin{aligned}
\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) &= 1 + i, \\
\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) &= \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right), \\
\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right) &= \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right).
\end{aligned}$$

5 The Complex Exponential Function

The real exponential function e^x (or $\exp x$) can be defined in several different ways. One such definition is by power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

(Recall here that the notation $n!$ denotes the product $1 \times 2 \times \cdots \times n$ and is read 'n factorial'). It is the case that the infinite sum above converges for all real values of x. What this means is that for any real value of our *input* x, as we add more and more of the terms from the infinite sum above we generate a list of numbers which get closer and closer to some value – this value we denote e^x. Different inputs will mean the sum converges to different answers. As an example let's consider the case when $x = 2$:

1 term:	1	$= 1.0000$	6 terms:	$1 + \cdots + \frac{32}{120}$	$\cong 7.2667$
2 terms:	$1 + 2$	$= 3.0000$	7 terms	$1 + \cdots + \frac{64}{720}$	$\cong 7.3556$
3 terms:	$1 + 2 + \frac{4}{2}$	$= 5.0000$	8 terms	$1 + \cdots + \frac{128}{5040}$	$\cong 7.3810$
4 terms:	$1 + \cdots + \frac{8}{6}$	$\cong 6.3333$	9 terms	$1 + \cdots + \frac{256}{40320}$	$\cong 7.3873$
5 terms:	$1 + \cdots + \frac{16}{24}$	$= 7.0000$	∞ terms	e^2	$\cong 7.3891$

This idea of a power series defining a function should not be too alien — it is likely that you have already seen that the infinite geometric progression

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

converges to $(1 - x)^{-1}$, at least when $|x| < 1$. This is another example of a power series defining a function.

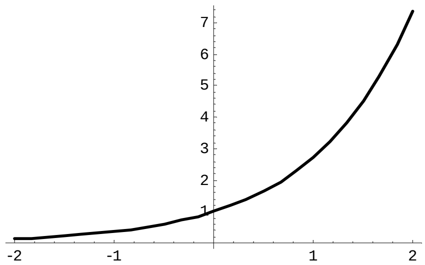
Proposition 16 *Let x be a real number. Then*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

converges to a real value which we shall denote as e^x. The function e^x has the following properties

$$\begin{array}{ll} (i) & \dfrac{d}{dx}e^x = e^x, \qquad e^0 = 1, \\ (ii) & e^{x+y} = e^x e^y \quad \textit{for any real } x, y. \\ (iii) & e^x > 0 \textit{ for any real } x. \end{array}$$

and a sketch of the exponential's graph is given below.



The graph of $y = e^x$.

That these properties hold true of e^x are discussed in more detail in the appendices at the end of this article.

- Property (i) also characterises the exponential function. That is, there is a unique real-valued function e^x which differentiates to itself, and which takes the value 1 at 0.
- Note that when $x = 1$ this gives us the identity

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

We can use either the power series definition, or one equivalent to property (i), to define the *complex* exponential function.

Proposition 17 *Let z be a complex number. Then*

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges to a complex value which we shall denote as e^z. The function e^z has the following properties

$$\begin{array}{ll} (i) & \dfrac{d}{dz}e^z = e^z, \qquad e^0 = 1, \\ (ii) & e^{z+w} = e^z e^w \quad \textit{for any complex } z, w, \\ (iii) & e^z \neq 0 \quad \textit{for any complex } z. \end{array}$$

Analytically we can differentiate complex functions in much the same way as we differentiate real functions. The product, quotient and chain rules apply in the usual way, and z^n has derivative nz^{n-1} for any integer n.

We can calculate e^z to greater and greater degrees of accuracy as before, by taking more and more terms in the series. For example to calculate e^{1+i} we see

$$\begin{array}{lll}
1 \text{ term:} & 1 & = 1.0000 \\
2 \text{ terms:} & 1+(1+i) & = 2.0000+1.0000i \\
3 \text{ terms:} & 1+(1+i)+\frac{2i}{2} & = 2.0000+2.0000i \\
4 \text{ terms:} & 1+\cdots+\frac{-2+2i}{6} & \cong 1.6667+2.3333i \\
5 \text{ terms:} & 1+\cdots+\frac{-4}{24} & \cong 1.5000+2.3333i \\
6 \text{ terms:} & 1+\cdots+\frac{-4-4i}{120} & \cong 1.4667+2.3000i \\
7 \text{ terms:} & 1+\cdots+\frac{-8i}{720} & \cong 1.4667+2.2889i \\
8 \text{ terms:} & 1+\cdots+\frac{8-8i}{5040} & \cong 1.4683+2.2873i \\
9 \text{ terms:} & 1+\cdots+\frac{16}{40320} & \cong 1.4687+2.2873i \\
\infty \text{ terms:} & e^{1+i} & \cong 1.4687+2.2874i
\end{array}$$

To close this section we introduce two functions related to the exponential function – namely *hyperbolic cosine* $\cosh z$ and *hyperbolic sine* $\sinh z$.

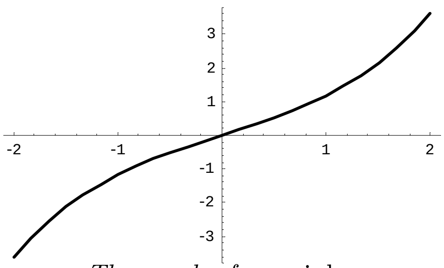
Definition 18 *Let z be a complex number. Then we define*

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \textit{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

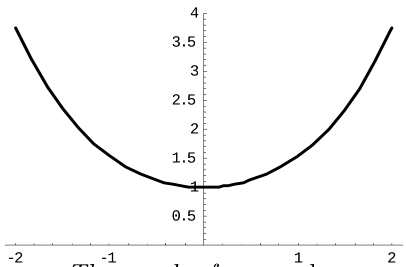
Corollary 19 *Hyperbolic sine and hyperbolic cosine have the following properties (which can easily be derived from the properties of the exponential function given in Proposition 17). For complex numbers z and w:*

$$\begin{array}{ll}
\textit{(i)} & \cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots + \dfrac{z^{2n}}{(2n)!} + \cdots \\
\textit{(ii)} & \sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots + \dfrac{z^{2n+1}}{(2n+1)!} + \cdots \\
\textit{(iii)} & \dfrac{d}{dz}\cosh z = \sinh z \quad \textit{and} \quad \dfrac{d}{dz}\sinh z = \cosh z, \\
\textit{(iv)} & \cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w, \\
\textit{(v)} & \sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w, \\
\textit{(vi)} & \cosh(-z) = \cosh z \quad \textit{and} \quad \sinh(-z) = -\sinh z.
\end{array}$$

and graphs of the sinh *and* cosh *are sketched below for real values of x*



The graph of $y = \sinh x$



The graph of $y = \cosh x$

6 The Complex Trigonometric Functions.

The real functions *sine* and *cosine* can similarly be defined by power series and other characterising properties. Note that these definitions give us sine and cosine of x *radians.*

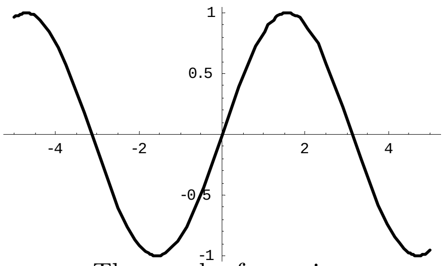
Proposition 20 *Let x be a real number. Then*

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

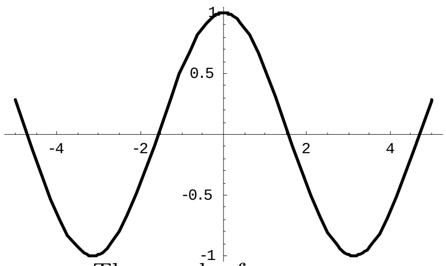
converge to real values which we shall denote as $\cos x$ *and* $\sin x$*. The functions* $\cos x$ *and* $\sin x$ *have the following properties*

$$\begin{array}{ll}
\textit{(i)} & \dfrac{d^2}{dx^2}\cos x=-\cos x, \qquad \cos 0=1, \quad \cos' 0=0,\\
\textit{(ii)} & \dfrac{d^2}{dx^2}\sin x=-\sin x, \qquad \sin 0=0, \quad \sin' 0=1,\\
\textit{(iii)} & \dfrac{d}{dx}\cos x=-\sin x, \quad \textit{and} \quad \dfrac{d}{dx}\sin x=\cos x,\\
\textit{(iv)} & -1\leq\cos x\leq 1 \quad \textit{and} \quad -1\leq\sin x\leq 1,\\
\textit{(v)} & \cos(-x)=\cos x \quad \textit{and} \quad \sin(-x)=-\sin x.
\end{array}$$

- Property (i) above characterises $\cos x$ and property (ii) characterises $\sin x$ – that is $\cos x$ and $\sin x$ are the unique real functions with these respective properties.



The graph of $y=\sin x$



The graph of $y=\cos x$

As before we can extend these power series to the complex numbers to define the complex trigonometric functions.

Proposition 21 *Let z be a complex number. Then the series*

$$1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots$$

converge to complex values which we shall denote as $\cos z$ *and* $\sin z$*. The functions* $\cos$ *and* $\sin$ *have the following properties*

$$\begin{array}{ll}
\textit{(i)} & \dfrac{d^2}{dz^2}\cos z=-\cos z, \qquad \cos 0=1, \quad \cos' 0=0,\\
\textit{(ii)} & \dfrac{d^2}{dz^2}\sin z=-\sin z, \qquad \sin 0=0, \quad \sin' 0=1,\\
\textit{(iii)} & \dfrac{d}{dz}\cos z=-\sin z, \quad \textit{and} \quad \dfrac{d}{dz}\sin z=\cos z,\\
\textit{(iv)} & \textit{Neither } \sin \textit{ nor } \cos \textit{ are bounded on the complex plane,}\\
\textit{(v)} & \cos(-z)=\cos z \quad \textit{and} \quad \sin(-z)=-\sin z.
\end{array}$$

Problem 22 *Prove that* $\cos^2 z + \sin^2 z = 1$ *for all complex numbers* z. *(Note that this does NOT imply that* $\cos z$ *and* $\sin z$ *have modulus less than or equal to 1.)*

Define

$$F(z) = \sin^2 z + \cos^2 z.$$

If we differentiate F using the previous proposition and the product rule we see

$$F'(z) = 2 \sin z \cos z + 2 \cos z \times (-\sin z) = 0.$$

As the derivative $F' = 0$ then F must be constant. We note that

$$F(0) = \sin^2 0 + \cos^2 0 = 0^2 + 1^2 = 1$$

and hence $F(z) = 1$ for all z.

Contrast this with:

Problem 23 *Prove that* $\cosh^2 - \sinh^2 z = 1$ *for all complex numbers* z.

Recall that

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

So using $e^z e^{-z} = e^{z-z} = e^0 = 1$ from Proposition 17

$$\begin{aligned} \cosh^2 z - \sinh^2 z &= \left[\frac{(e^z)^2 + 2e^z e^{-z} + (e^z)^2}{4}\right] - \left[\frac{(e^z)^2 - 2e^z e^{-z} + (e^{-z})^2}{4}\right] \\ &= \frac{4e^z e^{-z}}{4} = 1. \end{aligned}$$

Remark 24 *It is for these reasons that the functions* cosh *and* sinh *are called hyperbolic functions and the functions* sin *and* cos *are often referred to as the circular functions. From the first problem above we see that the point* $(\cos t, \sin t)$ *lies on the circle* $x^2 + y^2 = 1$. *As we vary* t *between* 0 *and* 2π *this point moves once anti-clockwise around the unit circle. In contrast the point* $(\cosh t, \sinh t)$ *lies on the curve* $x^2 - y^2 = 1$. *This is the equation of a hyperbola. As* t *varies through the reals then* $(\cosh t, \sinh t)$ *maps out all of the right branch of the hyperbola. We can obtain the left branch by varying the point* $(-\cosh t, \sinh t)$.

7 Identities

We prove here the fundamental identity relating the exponential and trigonometric functions.

Theorem 25 *Let* z *be a complex number. Then*

$$e^{iz} = \cos z + i \sin z.$$

Proof. Recalling the power series definitions of the exponential and trigonometric functions from Propositions 17 and 21 we see

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \cos z + i \sin z. \end{aligned}$$

■

- Note that $\cos z \neq \operatorname{Re} e^{iz}$ and $\sin z \neq \operatorname{Im} e^{iz}$ in general for complex z.
- When we put $z = \pi$ into this proposition we find

$$e^{i\pi} = -1$$

  which was a result first noted by Euler. Note also that the complex exponential function has period $2\pi i$. That is

$$e^{z+2\pi i} = e^z \quad \text{for all complex numbers } z.$$

- More generally when θ is a real number we see that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

  and so the polar form of a complex number is more commonly written as

$$z = re^{i\theta}$$

  rather than as $z = r\cos\theta + i\sin\theta$. Moreover in these terms, De Moivre's Theorem (see Theorem 12) states that

$$\left(e^{i\theta}\right)^n = e^{i(n\theta)}.$$

- If $z = x + iy$ then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

  and so

$$|e^z| = e^x \quad \text{and} \quad \arg e^z = y.$$

As a corollary to the previous theorem we can now express $\cos z$ and $\sin z$ in terms of the exponential. We note

Corollary 26 *Let z be a complex number. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\begin{aligned} \cosh z &= \cos iz \quad \textit{and} \quad i\sinh z = \sin iz \\ \cos z &= \cosh iz \quad \textit{and} \quad i\sin z = \sinh iz. \end{aligned}$$

Proof. As cos is even and sin is odd then

$$e^{iz} = \cos z + i\sin z$$

and

$$e^{-iz} = \cos z - i\sin z.$$

Solving for $\cos z$ and $\sin z$ from these simultaneous equations we arrive at the required expressions. The others are easily verified from our these new expressions for cos and sin and our previous ones for cosh and sinh . ■

8 Applications

We now turn these formula towards some applications and calculations.

Problem 27 *Let θ be a real number. Show that*

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Recall from De Moivre's Theorem that

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta.$$

Now if x and y are real then by the Binomial Theorem

$$(x+iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5.$$

Hence

$$\begin{aligned}
\cos 5\theta &= \operatorname{Re}(\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2 \\
&= (1+10+5)\cos^5\theta + (-10-10)\cos^3\theta + 5\cos\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.
\end{aligned}$$

Problem 28 *Let z be a complex number. Prove that*

$$\sin^4 z = \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8}.$$

Hence find the power series for $\sin^4 z$.

We have that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

So

$$\begin{aligned}
\sin^4 z &= \frac{1}{(2i)^4}\left(e^{iz} - e^{-iz}\right)^4 \\
&= \frac{1}{16}\left(e^{4iz} - 4e^{2iz} + 6 - 4e^{-2iz} + e^{-4iz}\right) \\
&= \frac{1}{16}\left(\left(e^{4iz} + e^{-4iz}\right) - 4\left(e^{2iz} + e^{-2iz}\right) + 6\right) \\
&= \frac{1}{16}\left(2\cos 4z - 8\cos 2z + 6\right) \\
&= \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8},
\end{aligned}$$

as required. Now $\sin^4 z$ has only even powers of z^{2n} in its power series. When $n > 0$ the coefficient of z^{2n} will then equal

$$\frac{1}{8} \times (-1)^n \frac{4^{2n}}{(2n)!} - \frac{1}{2} \times (-1)^n \frac{2^{2n}}{(2n)!} = (-1)^n \frac{2^{4n-3} - 2^{2n-1}}{(2n)!} z^{2n}$$

which we note is zero when $n = 1$. Also when $n = 0$ we see that the constant term is $1/8 - 1/2 + 3/8 = 0$. So the required power series is

$$\sin^4 z = \sum_{n=2}^{\infty} (-1)^n \frac{2^{4n-3} - 2^{2n-1}}{(2n)!} z^{2n}.$$

Problem 29 *Prove for any complex numbers z and w that*

$$\sin(z+w) = \sin z \cos w + \cos z \sin w.$$

Recalling the expressions for sin and cos from Corollary 26 we have

$$\begin{aligned}
\text{RHS} &= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right)+\left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) \\
&= \frac{2e^{iz}e^{iw}-2e^{-iz}e^{-iw}}{4i} \\
&= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} = \sin(z+w) = \text{LHS}.
\end{aligned}$$

Problem 30 *Prove that for complex z and w*

$$\sin(z+iw) = \sin z \cosh w + i\cos z \sinh w.$$

Use the previous problem recalling that $\cos(iw) = \cosh w$ and $\sin(iw) = i\sinh w$.

Problem 31 *Let x be a real number and n a natural number. Show that*

$$\sum_{k=0}^{n}\cos kx = \frac{\cos\frac{n}{2}x\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x} \quad \textit{and} \quad \sum_{k=0}^{n}\sin kx = \frac{\sin\frac{n}{2}x\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}$$

As $\cos kx + i\sin kx = \left(e^{ix}\right)^k$ then these sums are the real and imaginary parts of a geometric series, with first term 1, common ration e^{ix} and $n+1$ terms in total. So recalling

$$1+r+r^2+\cdots+r^n = \frac{r^{n+1}-1}{r-1},$$

we have

$$\begin{aligned}
\sum_{k=0}^{n}\left(e^{ix}\right)^k &= \frac{e^{(n+1)ix}-1}{e^{ix}-1} \\
&= \frac{e^{inx/2}\left(e^{(n+1)ix/2}-e^{-(n+1)ix/2}\right)}{e^{ix/2}-e^{-ix/2}} \\
&= e^{inx/2}\frac{2i\sin\frac{n+1}{2}x}{2i\sin\frac{1}{2}x} \\
&= \left(\cos\frac{nx}{2}+i\sin\frac{nx}{2}\right)\frac{\sin\frac{n+1}{2}x}{\sin\frac{1}{2}x}.
\end{aligned}$$

The results follow by taking real and imaginary parts.

9 Distance and Angles in the Complex Plane

Let $z = z_1 + iz_2$ and $w = w_1 + iw_2$ be two complex numbers. By Pythagoras' Theorem the distance between z and w as points in the complex plane equals

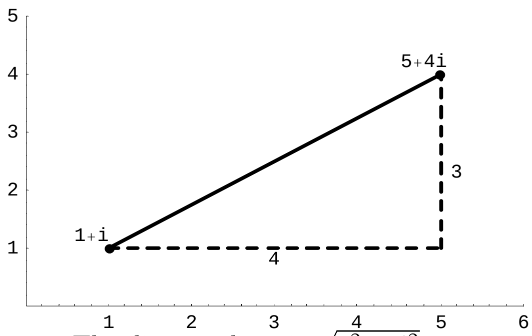
$$\begin{aligned} \text{distance} &= \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} \\ &= |(z_1 - w_1) + i(z_2 - w_2)| \\ &= |(z_1 + iz_2) - (w_1 + iw_2)| \\ &= |z - w|. \end{aligned}$$

Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $c = c_1 + ic_2$ be three points in the complex plane representing three points A, B and C. To calculate the angle $\measuredangle BAC$ as in the diagram we see

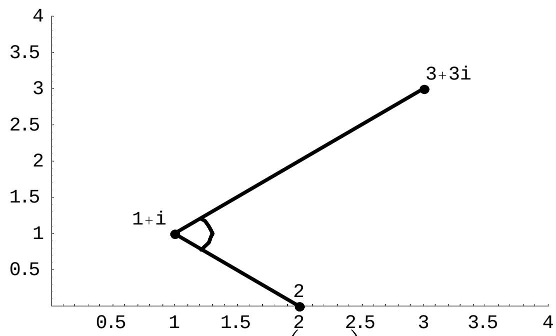
$$\measuredangle BAC = \arg(c - a) - \arg(b - a) = \arg\left(\frac{c - a}{b - a}\right).$$

Note that if in the diagram B and C we switched then we get the larger angle

$$\arg\left(\frac{c - a}{b - a}\right) = 2\pi - \measuredangle BAC.$$



The distance here is $\sqrt{3^2 + 4^2} = 5$



The angle here is $\arg\left(\frac{2+2i}{1-i}\right) = \arg 2i = \frac{1}{2}\pi$

Problem 32 *Find the smaller angle $\measuredangle BAC$ where $a = 1 + i$, $b = 3 + 2i$, and $c = 4 - 3i$.*

$$\begin{aligned} \measuredangle BAC &= \arg\left(\frac{b - a}{c - a}\right) \\ &= \arg\left(\frac{2 + i}{3 - 4i}\right) \\ &= \arg\left(\frac{(2 + i)(3 + 4i)}{3^2 + 4^2}\right) \\ &= \arg\left(\frac{2 + 11i}{25}\right) \\ &= \arctan\left(\frac{11}{2}\right) \cong 1.3909 \text{ radians.} \end{aligned}$$

10 A Selection of Geometric Theory

When using complex numbers to prove geometric theorems it is prudent to choose our complex co-ordinates so as to make the calculations as simple as possible. If we put co-ordinates on the plane we can choose

- where to put the origin;
- where the real and imaginary axes go;
- what unit length to use.

For example if we are asked to prove a theorem about a circle then we can take the centre of the circle as our origin. We can also choose our unit length to be that of the radius of the circle. If we have points on the circle to consider then we can take one of the points to be the point 1. However these choices (largely) use up all our 'degrees of freedom' and any other points need to be treated generally.

Similarly if we were considering a triangle then we could choose two of the vertices to be 0 and 1 but the other point (unless we know something special about the triangle, say that it is equilateral or isoceles) we need to treat as an arbitrary point z.

We now prove a selection of basic geometric facts. Here is a quick reminder of some identities (from Lecture 1) which will prove useful in their proofs.

$$\begin{aligned}
\operatorname{Re} z &= \frac{z+\overline{z}}{2}, \\
z\overline{z} &= |z|^2, \\
\cos\arg z &= \frac{\operatorname{Re} z}{|z|}.
\end{aligned}$$

Theorem 33 *(THE COSINE RULE). Let ABC be a triangle. Then*

$$|BC|^2 = |AB|^2 + |AC|^2 - 2\,|AB|\,|AC|\cos\hat{A}.$$

We can choose our co-ordinates in the plane so that A is at the origin and B is at 1. Let C be at the point z. So in terms of our co-ordinates:

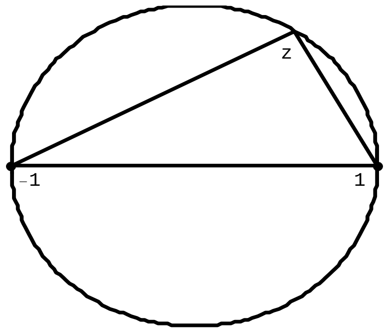
$$\begin{aligned}
|BC| &= |z-1|\,; \\
|AB| &= 1; \\
|AC| &= |z|\,; \\
\hat{A} &= \arg z.
\end{aligned}$$

So

$$\begin{aligned}
\text{RHS} &= |z|^2 + 1 - 2\,|z|\cos\arg z \\
&= z\overline{z} + 1 - 2\,|z| \times \frac{\operatorname{Re} z}{|z|} \\
&= z\overline{z} + 1 - 2 \times \frac{(z+\overline{z})}{2} \\
&= z\overline{z} + 1 - z - \overline{z} \\
&= (z-1)(\overline{z}-1) \\
&= |z-1|^2 = \text{LHS}.
\end{aligned}$$

Theorem 34 *The diameter of a circle subtends a right angle at the circumference.*

We can choose our co-ordinates in the plane so that the circle has unit radius with its centre at the origin and with the diameter in question having endpoints 1 and -1. Take an arbitrary point z in the complex plane – for the moment we won't assume z to be on the circumference.



From the diagram we see that below the diameter we want to show

$$\arg(-1-z) - \arg(1-z) = \frac{\pi}{2},$$

and above the diameter we wish to show that

$$\arg(-1-z) - \arg(1-z) = -\frac{\pi}{2}.$$

Recalling that $\arg(z/w) = \arg z - \arg w$ we see that we need to prove that

$$\arg\left(\frac{-1-z}{1-z}\right) = \pm\frac{\pi}{2}$$

or equivalently we wish to show that $(-1-z)/(1-z)$ is *purely* imaginary – i.e. it has no real part.

To say that a complex number w is purely imaginary is equivalent to saying that $w = -\overline{w}$ – i.e. that

$$\left(\frac{-1-z}{1-z}\right) = -\overline{\left(\frac{-1-z}{1-z}\right)}.$$

which is the same as saying

$$\frac{-1-z}{1-z} = \frac{1+\overline{z}}{1-\overline{z}}.$$

Multiplying up we see this is the same as

$$(-1-z)(1-\overline{z}) = (1+\overline{z})(1-z).$$

Expanding this becomes

$$-1-z+\overline{z}+z\overline{z} = 1+\overline{z}-z-z\overline{z}.$$

Rearranging this is the same as $z\overline{z} = 1$ but as $|z|^2 = z\overline{z}$ we see we must have

$$|z| = 1.$$

What we have now shown is in fact more than the required theorem. We have shown that the diameter subtends a right angle at a point on the circumference and subtends right angles nowhere else.

11 Transformations of the Complex Plane

We now describe some transformations of the complex plane and show how they can be written in terms of complex numbers.

Translations: A translation of the plane is one which takes the point (x, y) to the point $(x+a, y+b)$ where a and b are two real constants. In terms of complex co-ordinates this is the map $z \mapsto z + z_0$ where $z_0 = a + ib$.

Rotations: Consider rotation the plane about the origin anti-clockwise through an angle α. If we take an arbitrary point in polar form $re^{i\theta}$ then this will rotate to the point

$$re^{i(\theta+\alpha)} = re^{i\theta}e^{i\alpha}.$$

So this rotation is represented in complex co-ordinates as the map $z \mapsto ze^{i\alpha}$.

More generally, any rotation of $\mathbb{C}$, not necessarily about the origin has the form $z \mapsto az+b$ where $a, b \in \mathbb{C}$, with $|a| = 1$ and $a \neq 1$.

Reflections: We have already commented that $z \mapsto \overline{z}$ denotes reflection in the real axis.

More generally, any reflection about the origin has the form $z \mapsto a\overline{z} + b$ where $a, b \in \mathbb{C}$ and $|a| = 1$.

What we have listed here are the three types of *isometry* of $\mathbb{C}$. An isometry of $\mathbb{C}$ is a map $f : \mathbb{C} \to \mathbb{C}$ which preserves distance — that is for any two points z and w in $\mathbb{C}$ the distance between $f(z)$ and $f(w)$ equals the distance between a and b. Mathematically this means

$$|f(z) - f(w)| = |z - w|$$

for any complex numbers z and w.

Proposition 35 *Let $f : \mathbb{C} \to \mathbb{C}$ be an isometry. Then there exist complex numbers a and b with $|a| = 1$ such that*

$$f(z) = az + b \quad \text{or} \quad f(z) = a\overline{z} + b$$

for each $z \in \mathbb{C}$.

Problem 36 *Express in the form $f(z) = a\overline{z} + b$ reflection in the line $x + y = 1$.*

Solution one: Knowing from the proposition that the reflection has the form $f(z) = a\overline{z}+b$ we can find a and b by considering where two points go to. As 1 and i both lie on the line of reflection then they are both fixed. So

$$\begin{aligned} a1 + b &= a\overline{1} + b = 1, \\ -ai + b &= a\overline{i} + b = i. \end{aligned}$$

Substituting $b = 1 - a$ into the second equation we find

$$a = \frac{1-i}{1+i} = -i,$$

and $b = 1 + i$. Hence

$$f(z) = -i\overline{z} + 1 + i.$$

Solution two: We introduce as alternative method here — the idea of changing co-ordinates. We take a second set of complex co-ordinates in which the point $z = 1$ is the origin and for which the line of reflection is the real axis. The second complex co-ordinate w is related to the first co-ordinate z by

$$w = (1 + i)(z - 1).$$

For example when $z = 1$ then $w = 0$, when $z = i$ then $w = -2$, when $z = 2 - i$ then $w = 2$, when $z = 2 + i$ then $w = 2i$. The real axis for the w co-ordinate has equation $x + y = 1$ and the imaginary axis has equation $y = x - 1$ in terms of our original co-ordinates.

The point to all this is that as w's real axis is the line of reflection then the transformation we're interested in is given by $w \mapsto \overline{w}$ in the new co-ordinates.

Take then a point with complex co-ordinate z in our original co-ordinates system.

Its w-co-ordinate is $(1 + i)(z - 1)$ — note we haven't moved the point yet, we've just changed co-ordinates.

Now if we reflect the point we know the w-co-ordinate of the new point is $\overline{(1 + i)(z - 1)} = (1 - i)(\overline{z} - 1)$.

Finally to get from the w-co-ordinate of the image point to the z-co-ordinate we reverse the co-ordinate change to get

$$\frac{(1 - i)(\overline{z} - 1)}{1 + i} + 1 = -i(\overline{z} - 1) + 1 = -i\overline{z} + i + 1$$

as required.

12 Appendix 1 — Properties of the Exponential

In Proposition 17 we stated the following for any complex number z.

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots$$

converges to a complex value which we shall denote as e^z. The function e^z has the following properties

$$\begin{array}{ll} (i) & \dfrac{\mathrm{d}}{\mathrm{d}z}e^z=e^z, \qquad e^0=1, \\ (ii) & e^{z+w}=e^z e^w \text{ for any complex } z,w, \\ (iii) & e^z\neq 0 \text{ for any complex } z. \end{array}$$

For the moment we shall leave aside any convergence issues.

To prove property (i) we *assume* that we can differentiate a power series *term by term*. Then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}e^z &= \frac{\mathrm{d}}{\mathrm{d}z}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots\right) \\ &= 0+1+\frac{2z}{2!}+\frac{3z^2}{3!}+\cdots\frac{nz^{n-1}}{n!}+\cdots \\ &= 1+z+\frac{z^2}{2!}+\cdots\frac{z^{n-1}}{(n-1)!}+\cdots \\ &= e^z. \end{aligned}$$

We give two proofs of property (ii)

PROOF ONE: Let x be a complex variable and let y be a *constant* (but arbitrary) complex number. Consider the function

$$F(x)=e^{y+x}e^{y-x}.$$

If we differentiate F by the product and chain rules, and knowing that e^x differentiates to itself we have

$$F'(x)=e^{y+x}e^{y-x}+e^{y+x}\left(-e^{y-x}\right)=0$$

and so F is a constant function. But note that $F(y)=e^{2y}e^0=e^{2y}$. Hence we have

$$e^{y+x}e^{y-x}=e^{2y}.$$

Now set $x=(z-w)/2$ and $y=(z+w)/2$ and we arrive at required identity: $e^z e^w=e^{z+w}$.

PROOF TWO: If we multiply two (convergent) power series

$$\sum_{n=0}^{\infty}a_n t^n \quad\text{and}\quad \sum_{n=0}^{\infty}b_n t^n$$

we get another (convergent) power series

$$\sum_{n=0}^{\infty}c_n t^n \quad\text{where}\quad c_n=\sum_{k=0}^{n}a_k b_{n-k}.$$

Consider

$$\begin{aligned} e^{zt} &= \sum_{n=0}^{\infty}\frac{z^n}{n!}t^n \quad\text{so that}\quad a_n=\frac{z^n}{n!}, \\ e^{wt} &= \sum_{n=0}^{\infty}\frac{w^n}{n!}t^n \quad\text{so that}\quad b_n=\frac{w^n}{n!}. \end{aligned}$$

Then

$$\begin{aligned} c_n &= \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} z^k w^{n-k} \\ &= \frac{1}{n!} (z+w)^n , \end{aligned}$$

by the binomial theorem. So

$$e^{zt} e^{wt} = \sum_{n=0}^{\infty} \frac{z^n}{n!} t^n \sum_{n=0}^{\infty} \frac{w^n}{n!} t^n = \sum_{n=0}^{\infty} \frac{(w+z)^n}{n!} t^n = e^{(w+z)t}.$$

If we set $t = 1$ then we have the required result.

Property (iii), that $e^z \neq 0$ for all complex z follows from the fact that $e^z e^{-z} = 1$.

12.1 Appendix 2 – Power Series

We have assumed many properties of power series throughout this lecture which we state here though it is beyond the scope of the course to prove these facts rigorously.

As we have only been considering the power series of exponential, trigonometric and hyperbolic functions it would be reasonable, but incorrect, to think that all power series converge everywhere. This is far from the case.

Given a power series $\sum_{n=0}^{\infty} a_n z^n$ where the coefficients a_n are complex there is a real or infinite number R in the range $0 \leq R \leq \infty$ such that

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n z^n \quad \text{converges to some complex value when } |z| < R, \\ &\sum_{n=0}^{\infty} a_n z^n \quad \text{does not converge to a complex value when } |z| > R. \end{aligned}$$

What happens to the power series when $|z| = R$ depends very much on the individual power serises.

The number R is called the *radius of convergence* of the power series.

For the exponential, trigonometric and hyperbolic power series we have already seen that $R = \infty$.

For the geometric progression $\sum_{n=0}^{\infty} z^n$ this converges to $(1-z)^{-1}$ when $|z| < 1$ and does not converge when $|z| \geq 1$. So for this power series $R = 1$.

An important fact that we assumed in the previous appendix is that a power series can be differentiated term by term to give the derivative of the power series. So the derivative of $\sum_{n=0}^{\infty} a_n z^n$ equals $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and this will have the same radius of convergence as the original power series.